APPENDIX II.

Comparison Between Confluent Hypergeometric and Power Series Solutions

C.H. solution		Power series solution	
Eigen-values	Sherwood numbers and θ_m at $x = 0.05$	Eigen-values	Sherwood numbers and θ_m at $x = 0.05$
$Sh_w = 1$			
1.0	$Sh_o = 0.729651$	1.000843	$Sh_o = 0.731372$
4.656137		4.655097	
8.562020	$Sh_f = 2.69892$	8.561968	$Sh_f = 2.72262$
12.515834		12.515815	
16.487648	$\theta_m = 0.961772$	16.487604	$\theta_m = 0.951367$
20.468354		20.468229	
$Sh_w = 2$			
1.220150	$Sh_o = 1.13696$	1.220269	$Sh_o = 1.141064$
4.880902		4.875175	
8.749982	$Sh_f = 2.63479$	8.764895	$Sh_f = 2.656924$
12.677288		12.593927	
16.630025	$\theta_m = 0.9383585$	16.543274	$\theta_m = 0.937276$
20.596391		20.603113	
$Sh_w = 5$			
1.445974	$Sh_o = 1.68056$	1.445977	$Sh_o = 1.68089$
5.205500		5.205331	
9.075228	$Sh_f = 2.53139$	9.077524	$Sh_f = 2.53214$
12.987426		12.985926	
16.922820	$\theta_m = 0.903222$	16.873229	$\theta_m = 0.903221$
20.872707		20.829852	

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Trajectory Calculation of Particle Deposition in Deep Bed Filtration

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Part I. Model Formulation

The packed bed model developed by Payatakes, Tien, and Turian (1973a, 1973b) is used as a basis for the study of particle deposition in deep bed filtration. The size of the particulate matters present in the suspension is assumed to be sufficiently large for Brownian motion to be negligible, but small enough for straining to be unimportant. The prediction of the rate of particle deposition is based on the one-step trajectory approach. The collector is represented by a unit bed element of the porous media model and the particle trajectory equation is formulated to include the gravitational force, the hydrodynamic force and torque (including the correction for the presence of the unit cell wall), the London force (including the retardation effect, which is shown to be of primary importance under conditions usually met in deep bed filtration systems), and the electrokinetic force. Sample capture trajectories, including the limiting capture trajectories, are given.

Based on the limiting trajectories and the assumption of uniform particle distribution at the entrance of each unit cell, the number fractions (of suspended particles) impacted on each unit cell are determined and then used to calculate the fraction impacted on the entire unit collector and also the value of the filter coefficient for a clean bed. It is also shown how the capture trajectory calculation can be used to determine the local rate of deposition along the wall of a given unit cell.

SCOPE

The study reported in this paper represents the beginning phase of a long-range investigation whose objective is the development of a model for the filtration of a liquid suspension through a granular packed bed (or deep bed filtration). The recently proposed P-T-T (Payatakes, Tien, Turian) model for granular porous media is believed to provide a realistic basis for the modeling of the dynamic

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behavior of a granular filter bed, including both the history of the effluent quality and the pressure drop increase during the course of filtration. As a first step of the modeling work, this study is confined to the case of relatively clean filter bed and suspended particles with size large enough for Brownian motion to be insignificant, but small enough for straining to be negligible.

According to current practice, the dynamic behavior of deep bed filtration is described by a set of phenomenological equations involving two parameters, namely the filter coefficient for a clean bed λ_0 and the pressure gradient in a clean bed, as well as two functional expressions relating the change of the filter coefficient and pressure gradient with the extent of the particle deposition in the filter. The P-T-T model was used previously to calculate the pressure gradient in a clean bed (Payatakes, Tien and Turian, 1973a). In the present work a theoretical model based on the P-T-T porous media model is postulated, which enables the prediction of λ_0 in terms of the system and operating variables of the filtration process.

The theoretical prediction of λ_0 is based on the trajectory calculation, that is to say, the determination of the paths of individual suspended particles which flow through the filter. A similar approach has been used by previous investigators (Spielman and Goren, 1970; Yao et al., 1971; FitzPatrick, 1972; Spielman and FitzPatrick, 1973; FitzPatrick and Spielman, 1973; Spielman and Cukor, 1973). However, the present study gives a more complete consideration of the various forces acting on the suspended particles, including for the first time the retardation effect on the London force. Furthermore, unlike the previous studies, all of which are based on single collector models, this study is based on a more realistic porous media model, and, in principle, it can be extended for the estimation of the functional dependence of the filter coefficient and pressure gradient on the amount of deposited matter.

CONCLUSIONS AND SIGNIFICANCE

The unit collector concept for the study of particle deposition in deep bed filtration is introduced. In the present case, a unit bed element of the P-T-T model is the unit collector. Limiting capture trajectories are obtained from the integration of the particle trajectory equation formulated in this work, which takes into account the gravitational force, the hydrodynamic force and torque (including the correction for the presence of the unit cell wall), the molecular dispersion force, including the retardation effect, which is shown to be of primary importance under conditions usually encountered in deep bed filtration, and the double layer interaction force. Based on the limiting capture trajectories, the fractions impacted on each unit cell of the unit collector are calculated and then used to calculate the fraction impacted on the entire unit collector, as well as the filter coefficient for the filter bed. The method developed in this work also provides a means for the determination of the local rate of particle deposition along the wall of a given unit cell.

PHENOMENOLOGICAL EQUATIONS OF DEEP BED FILTRATION

The overall objective of the study reported here is the development of a complete model for deep bed filtration. In order to put the present work in perspective and to define the problems with which theoretical studies of deep bed filtration have to come to terms, a brief account on the phenomenological description of deep bed filtration will be given.

From practical considerations, the main features of the dynamic behavior of a deep bed filter are the history of the effluent quality and the pressure drop increase as filtration proceeds. The history of the effluent quality can be obtained from the integration of a system of two equations, namely, the conservation equation and the rate equation. The conservation equation can be written as (Herzig et al., 1970)

$$\frac{\partial \sigma}{\partial \tau} + v_s \left(\frac{\partial C}{\partial z} \right)_\tau = 0 \tag{1}$$

where τ is the time measured from the instant at which the suspension first reached the position under consideration and is defined by

$$\tau = t - \int_0^z \frac{\epsilon(\tilde{z}, t)}{v_s} d\tilde{z} \tag{2}$$

where t is the time measured from the instant when the suspension enters the filter and ϵ is the local porosity.

The porosity of the filter bed is a function of depth as well as of time. The relationship between ϵ and the specific deposit σ is given as

$$\epsilon = \epsilon_0 - \frac{\sigma}{(1 - \epsilon_d)} \tag{3}$$

where ϵ_0 is the initial porosity of the filter bed.

The filtration rate expression on a phenomenological basis has been studied extensively during the last two decades (Ives, 1960, 1969, 1971; Herzig et al., 1970; Mints, 1966; Mints et al., 1967; Mackrle et al., 1965). The commonly accepted expression, based on the assumption of no reentrainment, is given as follows (see Ives, 1960):

$$\left(\frac{\partial C}{\partial z} \right)_\tau = -\lambda C = -\lambda_0(\underline{x}_1) f_\lambda(\sigma; \underline{x}_2) C(z, \tau) \tag{4}$$

where λ_0 is the initial filter coefficient (that is, λ, corresponding to $\sigma = 0$), and f_λ is the functional relationship accounting for the change of the filter coefficient due to the particle deposition.

The pressure drop increase due to particle deposition within the filter bed is often expressed as follows:

$$\left(\frac{\partial P}{\partial z} \right)_\tau = \left(\frac{\partial P}{\partial z} \right)_{\tau=0} f_P(\sigma; \underline{x}_3) \tag{5}$$

where the subscript $\tau = 0$ refers to the situation prior to filtration (that is, $\sigma = 0$), and f_P is the functional relationship accounting for the pressure gradient increase. The overall pressure drop across the filter bed at time τ is given by

$$\Delta P = \Delta P(\tau) = \int_0^L \left(\frac{\partial P}{\partial z} \right)_\tau dz \tag{6}$$

The dynamic behavior of the filter can be obtained from the integration of Equations (1), (4), and (6) provided that $(\partial P/\partial z)_{\tau=0}$, $\lambda_0(\underline{x}_1)$ and the functional relationships $f_\lambda(\sigma; \underline{x}_2)$ and $f_P(\sigma; \underline{x}_3)$ are known. The purpose of any modeling work in deep bed filtration is the prediction of λ_0, $(\partial P/\partial z)_{\tau=0}$, f_λ, and f_P. The pressure gradient for a clean bed $(\partial P/\partial z)_{\tau=0}$ was calculated in Payatakes et al. (1973a). The present work is concerned with the prediction of λ_0.

TRAJECTORY APPROACH IN PARTICLE DEPOSITION

The particle deposition process occurring in deep bed filtration involves the transport and attachment of particulate matter from the liquid suspension upon the surfaces of the filter grains as the suspension flows through the filter. In principle, the extent of filtration can be determined by following the path of each and every particle of the suspension. With knowledge of the particle trajectories, and an understanding of the attachment mechanism of particulate matter to the filter grains once contact is made, one can determine the total amount of particulate matter retained inside the filter as well as that which escapes and, accordingly, the history of effluent quality. This is the basic idea behind the trajectory calculation in the study of the filtration process.

It is generally assumed that for trajectory calculations, the packed bed can be adequately represented by a single geometric entity. The use of a single spherical collector in trajectory work (Spielman and Goren, 1970; Yao et al., 1971; FitzPatrick, 1972; Spielman and FitzPatrick, 1973; FitzPatrick and Spielman, 1973; Spielman and Cukor, 1973) is a good example. In this work, it is assumed that the bed can be approximated by the model recently proposed by Payatakes et al. (1973a). With the use of this new model, a particle trajectory can be determined from Newton's law of motion based on the initial particle position (that is, that prior to its entry into the basic element of the porous medium), velocity field within the element, and the fields of the other relevant forces. From the trajectories and the initial particle concentration distribution, it is possible to calculate the fraction impacted, η_0, that is, the number fraction of particles that enter the unit bed element per unit time, which come into collision with its surface. Due to the magnitude of the molecular dispersion force (this will be discussed in detail later), it can be assumed that each collision results in capture (at least during the initial period of filtration), in other words, that $\hat{\eta}_0 = \eta_0$ where $\hat{\eta}_0$ is the fraction captured initially in a clean bed. In the present work the fraction impacted η_0 (as well as the fraction captured $\hat{\eta}_0$) are calculated taking into account deposition on the surface of a unit bed element, which represents a large number of grains and half pores of various sizes simultaneously (Payatakes et al., 1973a). This is a substantially more general concept than the definition of η_0 in all previous works, which is based on the deposition on a single grain alone.

One can relate η_0 with λ_0 by applying Equation (4) to a clean filter of thickness l (l is the length of periodicity of the P-T-T model) and requiring that the reduction of relative particulate concentration is η_0. The following expression is obtained:

$$\lambda_0 = \frac{1}{l} \ln \left(\frac{1}{1 - \hat{\eta}_0} \right) = \frac{1}{l} \ln \left(\frac{1}{1 - \eta_0} \right) \tag{7}$$

The problem of calculating λ_0 has thus been reduced to the calculation of l and η_0. l was calculated in Payatakes et al. (1973a). The estimation of η_0 is the objective of the present work.

PREVIOUS WORK

Because of the space limitation, only a brief account of previous work is given here and a more detailed discussion is given in the Supplement.*

The possibility of applying trajectory calculation to deep bed filtration was first mentioned by O'Melia and Stumm (1967), and implementation of this idea was given in the work of Yao (1968). Yao assumed that for particle deposition, the filter grains can be considered as a number of independent and identical spheres with a flow around the spheres approximated by the creeping flow solution around a sphere in a fluid of infinite extent. In Yao's calculation, only inertial, gravitational, and hydrodynamic forces were included.

Yao's use of the single sphere model (a single collector) in describing a granular filter bed is obviously oversimplified and perhaps unrealistic. His use of the Stokesian expression for the drag force acting on the suspended particles also requires correction when the suspended particles are close to the collector surfaces. However, if the drag force is corrected properly, the subsequent trajectory calculation in Yao's formulation would lead to the paradox of no particle deposition. This difficulty was first recognized by Spielman and Goren (1970) and underscores the importance of including surface forces, such as the molecular dispersion force, in the trajectory calculation even though these forces are short-ranged. More recent studies (FitzPatrick, 1972; FitzPatrick and Spielman, 1973) employed the use of Happel's (1958) porous media model for the trajectory calculation. A major difficulty in the use of Happel's model is the fact that the numerical integration necessary for the determination of a particle trajectory often has to be carried out beyond the domain of validity of the model. It should also be mentioned that all these studies were confined to the case of relatively clean filters. Neither the single sphere nor the Happel model can be used to study deposition of relatively large particles, or the effect of particle deposition on filtration efficiency and pressure drop increase, which remain central problems in the study of deep bed filtration.

SELECTION OF POROUS MEDIA MODEL

While a large number of models were proposed in the past, most of them are not suited for particle deposition studies because the basic premise on which these models were formulated was not concerned with the deposition process. For example, the capillary model was found to give filter coefficients at least two or three orders of magnitude less than experimental values (Payatakes, 1973); Payatakes et al., 1974a) principally because of the failure of the model to reflect the two-dimensional flow character-

* Supplement has been deposited as Document No. 02444 with the National Auxiliary Publications Service (NAPS), c/o Microfiche Publications, 305 E. 46 St., New York, N. Y. 10017 and may be obtained for $1.50 for microfiche or $5.00 for photocopies.

istics around the filter grains, which is important to the particle deposition. Porous media models of the single collector type, such as the single sphere in an infinite medium and the Happel (1958) model are useful for the modeling of deep bed filtration of small particles in a clean bed, but cannot be used for the modeling of filtration of large particles, or for the modeling of the effect of particle deposition on the filtration efficiency and the pressure gradient (Payatakes et al., 1974a). The main reason for these shortcomings are (1) the effect of the neighboring grains is entirely neglected in the case of the single sphere in an infinite medium model, and taken into account only to a certain extent and then indirectly in the case of Happel's model, and (2) the very important factors of the shape and size of the constrictions connecting the pores in a packed bed are neglected.

The P-T-T model proposed recently (Payatakes, 1973a) considers a packed bed as being comprised of a series of unit bed elements of thickness l. Each unit bed element, in turn, consists of a number of unit cells geometrically similar but different in size connected in parallel. A unit cell resembles a constricted tube with its axis parallel to the axis of the packed bed. The P-T-T model is specifically formulated to incorporate a number of features such as the converging-diverging flow, the presence of neighboring grains, the nonuniformity of pores, and the identification of flow passages, which, on physical considerations, are important to deep bed filtration.

According to the present formulation, the basic collector unit for the deposition process is one unit bed element and this will be called a unit collector to distinguish it from the single collector concept. The major difference is that a unit collector of the P-T-T model is formed from a large number of grains and half pores simultaneously. As shown in Payatakes et al. (1973a), the dimensions and size distribution of the unit cells comprising a unit bed element of a given filter bed can be determined from experimental measurements. From these data and the algorithm developed in Payatakes et al. (1973b) the flow field within each cell can be calculated and used for the trajectory calculation.

SYSTEM OF COORDINATES USED FOR THE PARTICLE TRAJECTORY CALCULATION

The cylindrical polar coordinates (z, r, θ) used for the solution of the flow through the unit cells (Payatakes et al., 1973b) is not suitable for the trajectory studies. A new system of coordinates, therefore, is introduced and its relation to the cylindrical polar coordinates is given.

Consider a spherical particle in the vicinity of the wall of a unit cell of the ith type, Figure 1. Let 0 be the center of the particle, and let $0P$ be normal to the wall. Let (x, w, y) be Cartesian coordinates with origin at P so that x is tangent to the wall and lies on the plane $(z, r, 0)$, w is tangent to the wall and is normal to the plane $(z, r, 0)$, and y is normal to the wall and lies on the plane $(z, r, 0)$. The new coordinate system (x, w, y) will be used for the description of the forces and torques acting on the particle, and it should be noted that it changes as the particle moves through the unit cell.

The cylindrical coordinates are related to the new system of coordinates with origin at P through the expressions,

$$z = z_P + x\cos\alpha(z_P) + y\sin\alpha(z_P) \tag{8}$$

$$r = +\sqrt{[r_w(z_P) + x\sin\alpha(z_P) - y\cos\alpha(z_P)]^2 + w^2} \tag{9}$$

$$\theta = \tan^{-1}\left(\frac{-w}{r_w(z_P)}\right) \tag{10}$$

where z_P is the value of z at P, and α is the angle formed by the z-axis and the tangent on the wall at point P. It is given by

$$\alpha = \tan^{-1}\left(\frac{dr_w}{dz}\right) = \tan^{-1}\left(\frac{dr_w^*}{dz^*}\right) = \tan^{-1}\Phi(z^*) \tag{11}$$

from which it follows that ($\alpha \leq 0$ for $1/2 \leq z^* \leq 1$) and ($\alpha \geq 0$ for $1 \leq z^* \leq 3/2$). Other pertinent geometric relationships are shown in Figure 1.

For trajectory calculations, let ζ_2 be the distance between wall and particle center, measured from the wall, and let ζ_1 be the arc length BP, measured from B (Figure 1a). ζ_1 and ζ_2 describe the position of the center of the particle in the x and y directions, and they will be the variables in terms of which the particle trajectory will be described. Given z_P (and $z_P^* = z_P/h_i$), the corresponding value of ζ_1, ζ_{1P} is derived from

$$\zeta_{1P} = \int_{\frac{1}{2}h_i}^{z_P} \frac{dz}{\cos\alpha(z)} = h_i \int_{\frac{1}{2}}^{z_P^*} \sqrt{1 + \Phi^2(z^*)}\, dz^* \tag{12}$$

$\Phi(z^*)$ is given as (Payatakes et al., 1973b)

$$\Phi(z^*) = 8\ (r_2^* - r_3^*)(z^* - 1) \quad \text{for} \quad \frac{1}{2} \leq z^* \leq \frac{3}{2} \tag{13}$$

Substituting Equation (13) into Equation (12), one has

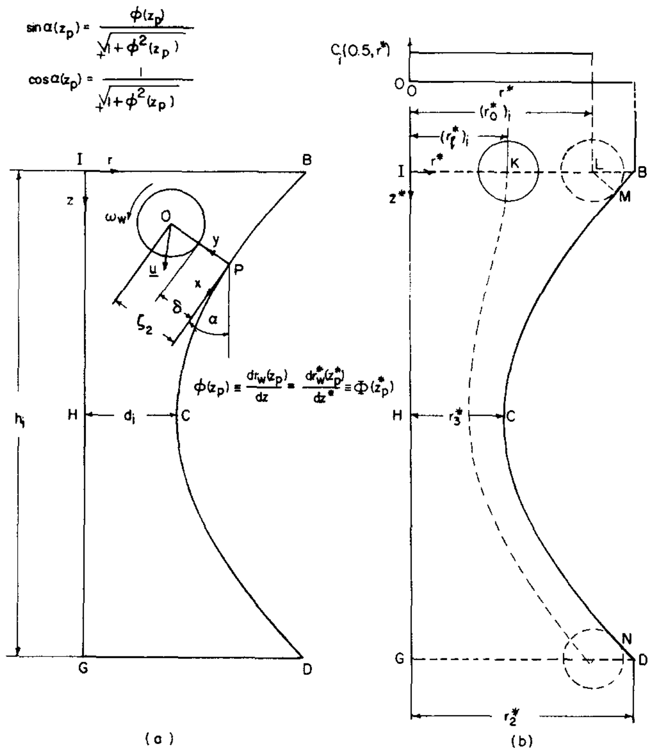

Fig. 1. (a) Particle moving in the vicinity of the wall of a dimensional unit cell of the ith type. Not to scale. (b) Schematic representation of limiting trajectory and of particle position closest to the wall at the entrance of the dimensionless unit cell (modified dimensions). Not to scale.

$$\zeta_{1P} = \frac{h_i}{16(r_2^* - r_3^*)} [\ln(\Phi_P + \sqrt{1+\Phi_P^2}) + \Phi_P\sqrt{1+\Phi_P^2} + \ln(\Phi_0 + \sqrt{1+\Phi_0^2}) + \Phi_0\sqrt{1+\Phi_0^2}] \quad (14)$$

where

$$\Phi_P = \Phi(z_P^*) = 8(r_2^* - r_3^*)(z_P^* - 1), \quad (15)$$

and

$$\Phi_0 = 4(r_2^* - r_3^*) \quad (16)$$

FORCES AND TORQUES ACTING ON THE PARTICLE

The particle is moving on the plane (x, y) and is also rotating about the axis parallel to the w-axis and passing through its center, 0. In order to determine the trajectory of the particle, one has to estimate all the forces and torques acting on it. From classical mechanics, one has

$$\underline{F_G} + \underline{F_L}^{ret} + \underline{F_E} + \underline{F_D} + \underline{F_I} = \underline{0} \quad (17)$$

$$\underline{T_G} + \underline{T_L}^{ret} + \underline{T_E} + \underline{T_D} + \underline{T_I} = \underline{0} \quad (18)$$

where $\underline{F_G}$, $\underline{F_L}^{ret}$, $\underline{F_E}$, $\underline{F_D}$ and $\underline{F_I}$ are the gravitational, London, electrokinetic, drag and inertia force vectors, respectively, and $\underline{T_G}$, $\underline{T_L}^{ret}$, $\underline{T_E}$, $\underline{T_D}$ and $\underline{T_I}$ are the corresponding torques. The Brownian force has not been included; therefore, the results are valid only when Brownian effects are negligible. This is usually true for particles with $a_p > 1\mu$, (for more details see Part II).

The procedures involved in the estimation of the components of the forces and torques appearing in Equations (17) and (18) are very lengthy and therefore are not included here. A detailed description is given in the Supplement.* The conditions under which these calculations are valid are also given there. The results are summarized below.

$$F_{Ix} = F_{Lx}^{ret} = F_{Ex} = 0,$$
$$T_{Ix} = T_{Gx} = T_{Lx}^{ret} = T_{Ex} = T_{Dx} = 0 \quad (19)$$

$$F_{Gx} = \frac{4}{3}\pi a_p^3(\rho_p - \rho) g \cos\alpha \quad (20)$$

$$F_{Dx} = -6\pi\mu a_p [u_x f_x^t(\delta^+) - a_p\omega_w f_x^r(\delta^+) - B_i\zeta_2 f_{1x}^m(\delta^+) - D_i\zeta_2^2 f_{2x}^m(\delta^+)] \quad (21)$$

$$F_{Iw} = F_{Gw} = F_{Lw}^{ret} = F_{Ew} = F_{Dw} = 0,$$
$$T_{Iw} = T_{Gw} = T_{Lw}^{ret} = T_{Ew} = 0 \quad (22)$$

$$T_{Dw} = 8\pi\mu a_p^2 [u_x g_w^t(\delta^+) - a_p\omega_w g_w^r(\delta^+) + B_i a_p g_{1w}^m(\delta^+) + D_i a_p \zeta_2 g_{2w}^m(\delta^+)] \quad (23)$$

$$F_{Iy} = 0, \quad T_{Iy} = T_{Gy} = T_{Ly}^{ret} = T_{Ey} = T_{Dy} = 0 \quad (24)$$

$$F_{Gy} = \frac{4}{3}\pi a_p^3(\rho_p - \rho) g \sin\alpha \quad (25)$$

$$F_{Ly}^{ret} = -\frac{2}{3}\frac{H}{a_p}\frac{\alpha_{s,p}(\delta; a_p, \lambda_e)}{(\delta^+)^2(2+\delta^+)^2} \quad (26)$$

$$F_{Ey} = \frac{\tilde{\epsilon}\, a_p\kappa(\psi^2_{01} + \psi^2_{02})}{2}\left[\frac{2\psi_{01}\psi_{02}}{(\psi^2_{01} + \psi^2_{02})} - e^{-\kappa\delta}\right]\frac{e^{-\kappa\delta}}{1 - e^{-2\kappa\delta}} \quad (27)$$

$$F_{Dy} = -6\pi\mu a_p [u_y f_y^t(\delta^+) + A_i\zeta_2^2 f_y^m(\delta^+)] \quad (28)$$

* See footnote on page 891.

where

$$\delta = \zeta_2 - a_p, \quad \delta^+ = \frac{\delta}{a_p} = \zeta_2^+ - 1 \quad (29)$$

$$\alpha_{s,p} \cong \frac{1}{1 + 0.620725p + 0.075159p^2} \quad \text{for} \quad 0 \leq p = \frac{2\pi\delta}{\lambda_e} \leq 3$$
$$\cong \frac{1.024172}{p} - \frac{0.714228}{p^2} + \frac{0.555262}{p^3} \quad \text{for} \quad 3 \leq p = \frac{2\pi\delta}{\lambda_e} < \infty \quad (30)$$

$$\kappa = \sqrt{\frac{4\pi e^2}{\tilde{\epsilon} kT}\sum_j m_j z_j^2} \quad (31)$$

The undisturbed fluid velocity inside a unit cell of the ith type is given by (see Supplement)

$$v_x = B_i y + D_i y^2, \quad v_y = -A_i y^2 \quad (32)$$

where the coefficients A_i, B_i, D_i vary along the wall of the unit cell. Calculated values of the universal functions appearing in the above equations are given in Tables 1 and 2 of the Supplement.

CALCULATION OF PARTICLE TRANSLATIONAL VELOCITY AND ANGULAR VELOCITY

Calculation of u_x and ω_w

Substituting the x components of the force terms for Equations (19), (20), and (21) into Equation (17) and the w components of the torque terms from Equations (22) and (23) into Equation (18), one obtains a system of two equations which can be solved for u_x and ω_x to give

$$u_x = \frac{d\zeta_1}{dt} = F_1(\delta^+)B_i\zeta_2 + F_2(\delta^+)D_i\zeta_2^2 + F_3(\delta^+)\frac{2(\rho_p - \rho)a_p^2 g}{9\mu}\cos\alpha \quad (33)$$

$$\omega_w = G_1(\delta^+)B_i + G_2(\delta^+)D_i\zeta_2 + G_3(\delta^+)\frac{2(\rho_p - \rho)a_p g}{9\mu}\cos\alpha \quad (34)$$

$$F_1 = \frac{\frac{1}{(1+\delta^+)} f_x^r g^m{}_{1w} + f^m{}_{1x} g_w^r}{f_x^t g_w^r - f_x^r g_w^t},$$

$$F_2 = \frac{\frac{1}{(1+\delta^+)} f_x^r g^m{}_{2w} + f^m{}_{2x} g_w^r}{f_x^t g_w^r - f_x^r g_w^t},$$

$$F_3 = \frac{g_w^r}{f_x^t g_w^r - f_x^r g_w^t} \quad (35)$$

$$G_1 = \frac{(1+\delta^+) f^m{}_{1x} g_w^t + f_x^t g^m{}_{1w}}{f_x^t g_w^r - f_x^r g_w^t},$$

$$G_2 = \frac{(1+\delta^+) f^m{}_{2x} g_w^t + f_x^t g^m{}_{2w}}{f_x^t g_w^r - f_x^r g_w^t},$$

$$G_3 = \frac{g_w^t}{f_x^t g_w^r - f_x^r g_w^t} \quad (36)$$

Calculated values of F_1, F_2, F_3, G_1, G_2, and G_3 are given in Table 3 of the Supplement.

Calculation of u_y

Substituting the y components of the force terms from Equations (24) to (28) into Equation (17), one obtains an equation which can be solved for u_y to give

$$u_y = \frac{d\zeta_2}{dt} = \frac{1}{f_y^t}\left\{ - A_i\zeta_2^2 f_y^m(\delta^+) + \frac{2(\rho_p-\rho)a_p^2 g}{9\mu}\sin\alpha + \frac{\widetilde{\epsilon\kappa}(\psi^2_{01}+\psi^2_{02})}{12\pi\mu}\left[\frac{2\psi_{01}\psi_{02}}{(\psi^2_{01}+\psi^2_{02})} - e^{-\kappa\delta}\right]\frac{e^{-\kappa\delta}}{1-e^{-2\kappa\delta}} - \frac{H}{9\pi\mu a_p^2}\,\frac{\alpha_{s,p}(\delta;a_p,\lambda_e)}{(\delta^+)^2(2+\delta^+)^2}\right\} \quad (37)$$

PARTICLE TRAJECTORY EQUATION

Using Equations (33) and (37), one obtains

$$\frac{d\zeta_2^+}{d\zeta_1^+} = \frac{-A_i^+\zeta_2^{+2}f_y^m + N_G\sin\alpha + N_{E1}(N_{E2}-e^{-N_{DL}\delta^+})\dfrac{e^{-N_{DL}\delta^+}}{(1-e^{-2N_{DL}\delta^+})} - \dfrac{N_{Lo}\alpha_{s,p}(\delta^+;N_{\text{Ret}})}{\delta^{+2}(2+\delta^+)^2}}{B_i^+\zeta_2^+F_4 + D_i^+\zeta_2^{+2}F_5 + N_G F_6\cos\alpha} \quad (38)$$

where

$$\zeta_1^+ = \frac{\zeta_1}{a_p},\quad \zeta_2^+ = \frac{\zeta_2}{a_p} \quad (39)$$

$$N_G = \frac{2(\rho_p-\rho)a_p^2 g}{9\mu v_s},\quad N_{E1} = \frac{\widetilde{\epsilon\kappa}(\psi^2_{01}+\psi^2_{02})}{12\pi\mu v_s},$$

$$N_{E2} = \frac{2\psi_{01}\psi_{02}}{(\psi^2_{01}+\psi^2_{02})} \quad (40)$$

$$N_{DL} = \kappa a_p,\quad N_{Lo} = \frac{H}{9\pi\mu a_p^2 v_s},\quad N_{\text{Ret}} = \frac{2\pi a_p}{\lambda_e} \quad (41)$$

$$A_i^+ = \frac{(v_0)_i}{v_s}\left(\frac{<d_g>}{h_i}N_{RS}\right)^2 A^*,\quad A^* = \frac{A_i h_i^2}{(v_0)_i},\quad i = 1,\ldots,I_c \quad (42)$$

$$B_i^+ = \frac{(v_0)_i}{v_s}\left(\frac{<d_g>}{h_i}N_{RS}\right)B^*,\quad B^* = \frac{B_i h_i}{(v_0)_i},\quad i = 1,\ldots,I_c \quad (43)$$

$$D_i^+ = \frac{(v_0)_i}{v_s}\left(\frac{<d_g>}{h_i}N_{RS}\right)^2 D^*,\quad D^* = \frac{D_i h_i^2}{(v_0)_i},\quad i = 1,\ldots,I_c \quad (44)$$

with

$$N_{RS} = \frac{a_p}{<h>} = \frac{a_p}{<d_g>} \quad (45)$$

$$F_4 = F_1 f_y^t = \frac{\dfrac{1}{(1+\delta^+)}f_x^r g^m_{1w} + f^m_{1x}g_w^r}{f_x^t g_w^r - f_x^r g_w^t}f_y^t \quad (46)$$

$$F_5 = F_2 f_y^t = \frac{\dfrac{1}{(1+\delta^+)}f_x^r g^m_{2w} + f^m_{2x}g_w^r}{f_x^t g_w^r - f_x^r g_w^t}f_y^t \quad (47)$$

$$F_6 = F_3 f_y^t = \frac{f_y^t g_w^r}{f_x^t g_w^r - f_x^r g_w^t} \quad (48)$$

Values of the universal functions F_4, F_5, and F_6, based on the values in Table 1 and 3 of the Supplement, are given in Table 4 of the Supplement and are plotted in Figure 2. For computational purposes these values have been interpolated to obtain approximate analytical expressions for F_4, F_5, and F_6, which are given in the Supplement. A similar expression was developed for f_y^m and is also given in the Supplement.

Assuming creeping flow conditions, the variables A^*, B^*, and D^* are the same for unit cells of all types, and they depend on ζ_1^* alone, where

$$\zeta_1^* = \frac{\zeta_1}{h_i} = \zeta_1^+\frac{a_p}{h_i} = \zeta_1^+(a_p^*)_i \quad (49)$$

A^*, B^*, and D^* are also calculated in the Supplement. The forms of A^*, B^*, and D^* are shown in Figure 3 for a unit cell with $r_1^* = r_2^* = 0.402$, and $r_3^* = 0.1685$ for creeping flow conditions.

SIMPLIFICATIONS FOR THE CALCULATION OF PARTICLE TRAJECTORIES

Two major simplifications will be introduced to the trajectory equation [Equation (38)]. They are:

1. The flow through each unit cell is assumed to be creeping. For the flow rates encountered in deep bed filtration, this is a good approximation. In the general case, the P-T-T model postulates that for a given superficial velocity v_s the pressure differences at the ends of all types of unit cells belonging to the same unit bed element are the same, and that the flow rates through unit cells of different types are different. Therefore, one needs to determine the flow velocity through the dimensionless unit cell $\underline{v}^*$ for I_c different values of $(N_{Re})_i$. However, with the assumption of creeping flow, the velocity vector $\underline{v}^*$ needs be determined only for Reynolds number equal to zero. Furthermore, the term $(v_0)_i/v_s$ appearing in the trajectory equation [Equations (42 to 44)] is given by

$$\frac{(v_0)_i}{v_s} = \frac{8r_3^* h_i}{\pi N_c <d_c^3>}\left(\frac{r_3^*}{r_1^*}\right)^2 \quad (50)$$

2. A modification of the geometry of the extended dimensionless unit cell is made as follows: The sections preceding and following the unit cell are assumed to be straight cylindrical tubes instead of following the periodic nature required in the original P-T-T model of granular

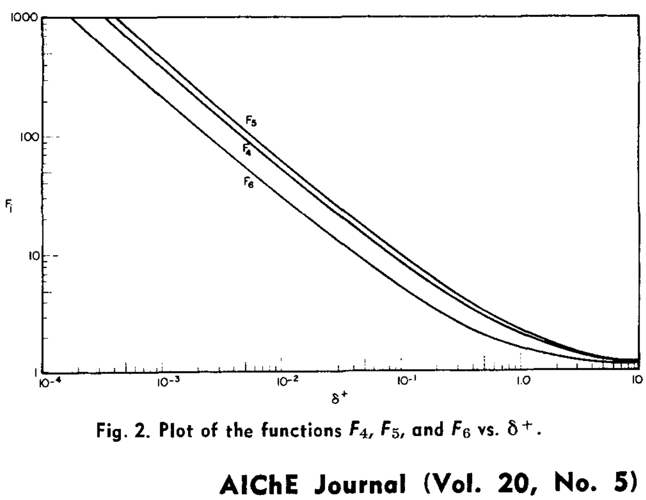

Fig. 2. Plot of the functions F_4, F_5, and F_6 vs. δ^+.

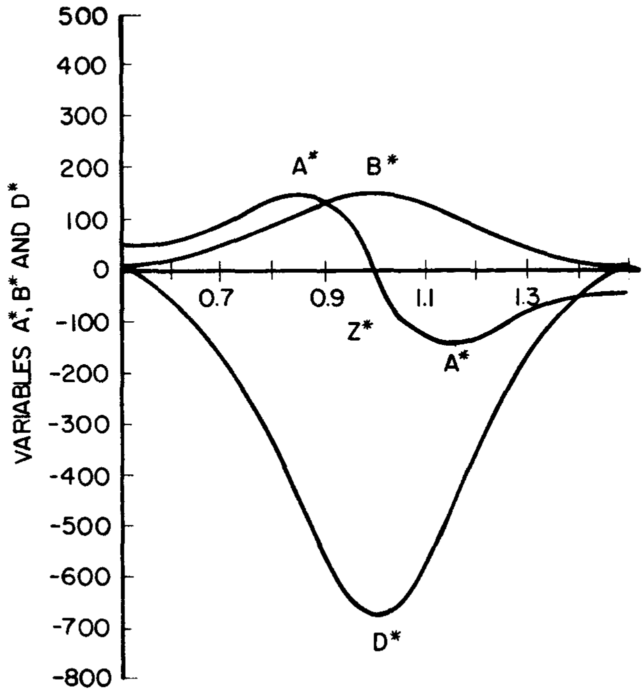

Fig. 3. Plot of the variables used for the description of the flow in the vicinity of the unit cell wall.

beds (see Figure 1b). This was judged necessary because the particle trajectories leading to capture originate from the immediate vicinity of the point B (Figure 1). A rigorous calculation would require that the effect of the presence of the wall section preceding the unit cell be taken into account, and this is an exceedingly difficult problem. Considering the fact that the regions of the unit cell at $z^* = 1/2$ and $z^* = 2/3$ (entrance and outlet) are the ones with minimal resemblance to the real porous media (see Figure 1), it was decided that a modification which would permit the omission of the effect of the wall of the entrance extension is preferable. One, however, has to recognize the fact that this modification results in a somewhat different flow field within the domain of interest. This velocity field corresponds to a nonuniform periodically constricted tube rather than a uniform periodically constricted tube (Payatakes et al., 1973b).

METHOD OF INTEGRATION OF THE TRAJECTORY EQUATION

A method for the integration of the particle trajectory equation [Equation (38)] which can be used to obtain not only the limiting trajectory (as previous methods), but also capture trajectories for any point on the wall and also the local rate of deposition on the unit cell wall is developed and described in the Supplement.

SAMPLE TRAJECTORY CALCULATIONS

A number of capture trajectories based on the data in Table 1 and the corresponding flow data [obtained using the algorithm developed in Payatakes et al. (1973b)], were calculated and plotted in Figure 4. These calculations correspond to two typical cases, one in which the double layer interaction force F_{Ey} is dominated at all separations by the London force F_{Ly}^{ret} (see Figure 2 in Supplement), and one in which F_{Ey} dominates F_{Ly}^{ret} beyond a critical distance from the unit cell wall (Figure 3 in Supplement).

In both cases, calculated trajectories which lead to capture at points on the wall of the lower half of the unit cell $(1 < z^* \leq 3/2)$ almost coincide in the upper half of the cell $(1/2 \leq z^* \leq 1)$, which implies that most of the particles collected are captured by the upper half of the cell. It should also be noted that in both situations a segment of the limiting trajectory is almost parallel to the lower half of the unit cell wall and that in the case when the repulsive electrokinetic force dominates the London force the entire limiting trajectory is almost parallel to the cell wall except for the final segment, which is nearly normal to the wall. In the latter case, the distance of the limiting trajectory from the wall for any given value of ζ_1^+ can be determined, to a very good approximation, by determining the value of δ^+ for which the sum $F_{Ey} + F_{Ly}^{ret} + F_{Gy} + F_{Dy} = 0$, namely, $\tilde{\delta}^+_{cr}(\zeta_1^+)$ or in other words, the value of δ^+ for which the numerator of the expression on the right-hand side of Equation (38) becomes zero. Further simplifications can be achieved if one capitalizes on the fact that since the particle is moving parallel to the wall F_{Dy} must be very small and the fact that at very small separations F_{Gy} is negligibly small compared to F_{Ly}^{ret}. Hence, the distance under consideration can be determined, approximately, by determining the critical value of δ^+ for which $F_{Ey} + F_{Ly}^{ret} = 0$, δ^+_{cr}. For

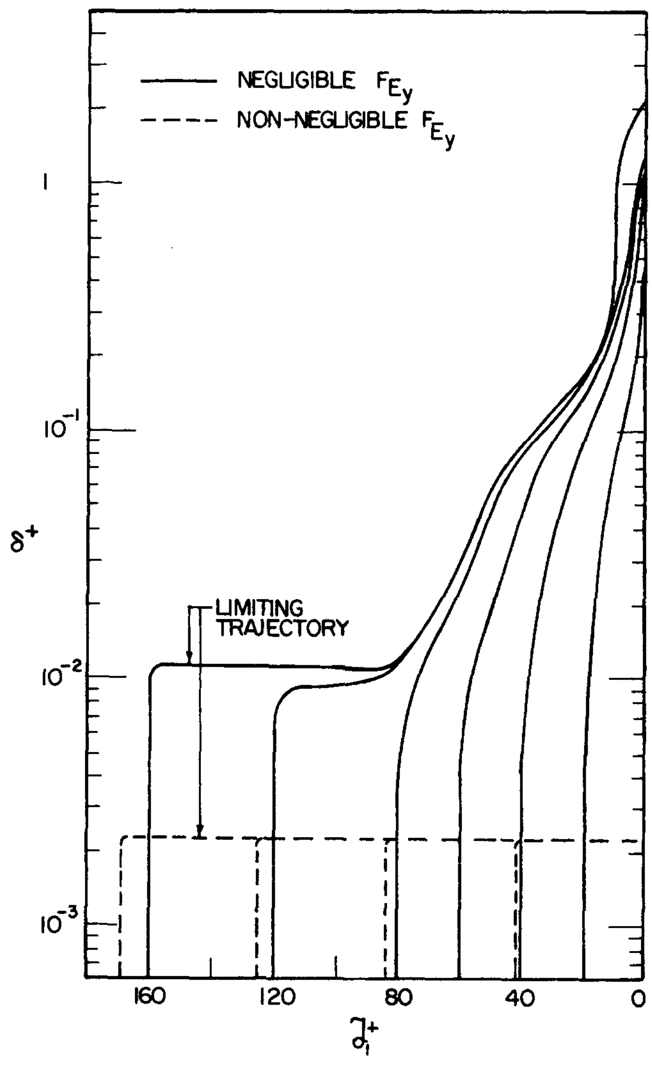

Fig. 4. Capture trajectories for the conditions in Table 1.

TABLE 1. PARAMETER VALUES ON WHICH THE CALCULATED TRAJECTORIES IN FIGURE 4 ARE BASED

Case Parameter	Negligible F_{Ey} Value	Non-negligible F_{Ey} Value*
r_1^*	0.402	0.3975
r_2^*	0.402	0.3975
r_3^*	0.1685	0.1760
h_i	714μ	720μ
$(v_0)_i$	0.24274 cm s^{-1}	0.056172 cm s^{-1}
N_c	178 cm^{-2}	180 cm^{-2}
v_s	0.1358 cm s^{-1}	0.03 cm s^{-1}
T	25.0°C	20.0°C
ρ	0.99708 g cm^{-3}	1.0 g cm^{-3}
μ	0.008937 poise	0.010050 poise
a_p	5μ	4.75μ
ρ_p	1.5 g cm^{-3}	1.06 g cm^{-3}
$\tilde{\epsilon}$	81	81
ψ_{01}	−30 mV	−70 mV
ψ_{02}	−8 mV	−50 mV
κ	2.8×10^6 cm^{-1}	5.9×10^4 cm^{-1}
H	5×10^{-13} erg	1.01×10^{-13} erg
λ_e	10^{-5} cm	10^{-5} cm
$(N_{Re})_i$	1.934	0.402427
N_G	0.022584	9.78831×10^{-3}
N_{E1}	53.0952	34.5707
N_{E2}	0.497925	0.945946
N_{DL}	1400	28.025
N_{Lo}	5.82836×10^{-5}	5.25115×10^{-5}
N_{RS}	7.0028×10^{-3}	6.59722×10^{-3}
N_{Ret}	314.159	298.451

* These values correspond to a set of experimental conditions used by FitzPatrick (1972), except for the value of v_s.

example, in the particular case of the conditions in Table 1, the separation value for which $F_{Ey} + F_{Ly}^{ret} = 0$ is $\delta^+_{cr} = 2.232 \times 10^{-3}$. The distance of the corresponding limiting trajectory, shown in Figure 4, from the wall at $z^* = 1.45$, 1.25, 1.0, 0.75, and 0.50 is $\delta^+ = 2.236 \times 10^{-3}$, 2.235×10^{-3}, 2.232×10^{-3}, 2.234×10^{-3} and 2.234×10^{-3}, respectively. Accordingly, the value of δ^+_{cr} can be used directly to calculate the location of the limiting trajectory in cases where $(F_{Ey} + F_{Ly}^{ret})$ displays a strong maximum. In the event that there is more than one value of δ^+ for which $(F_{Ey} + F_{Ly}^{ret})$ becomes zero, obviously the smallest one is to be considered. This approximate approach is strictly applicable to cases with dominant repulsive electrokinetic forces. In physical terms, the limiting trajectory shown in Figure 4 should be interpreted as follows. When strong repulsive electrokinetic forces which dominate the attractive London force beyond a critical distance are present, a particle can be captured by the wall only if its original position is close enough to the wall so that there is no need for its penetration through the region of dominant repulsive forces for its capture. This leads to a very small value of λ_0, which is in qualitative agreement with experimental observations (FitzPatrick, 1972). However, in view of the fact that $\delta^+ = 2.23 \times 10^{-3}$ with $a_p = 4.75\mu$ corresponds to a dimensional separation of $\delta = 106$Å, one has to question, in this case, the validity of the assumption of a smooth wall and of a smooth suspended particle surface. Materials encountered in practice possess surface anomalies which exceed by far 100Å, and, therefore, one cannot expect the theoretical trajectory approach to produce reliable predictions unless a way is devised to incorporate in the calculation the effect of the surface roughness. Furthermore, one may argue that in the case of non-negligible repulsive double layer force, one cannot assume that particles exist at the unit cell entrance at distances from the wall smaller than the critical one since they would have to overcome the potential energy barrier to reach such positions. In this case, the theoretical prediction is $\eta_0 = 0$ and $\lambda_0 = 0$.

The strong influence exhibited by the manner of combination of the double layer force and the molecular dispersion force on the value of η_0 as shown above is also consistent with the recent study of Spielman and Cukor (1973). These authors extended the work of FitzPatrick (1972) and included the double layer force in the trajectory calculation although they also did not include the retardation effect (see Part II for magnitude of this effect). These authors show that collection of particles can be attained by a secondary minimum. In the case of their model, capture by a secondary minimum is possible at a small distance immediately behind the rear stagnation point of the spherical collector, where a spherical particle can be held at equilibrium (by balancing forces normal to the collector wall) due to the absence of a net tangential force. In the framework of the model developed in this work, capture by a secondary minimum is not possible due to the different configuration of the collector wall. Indeed, a particle at a small distance from the wall of a unit cell, where it experiences a zero net force in the direction normal to the wall, always experiences a positive net tangential force that drives it to the exit of the cell.

CALCULATION OF THE IMPACTED FRACTIONS AND OF THE INITIAL FILTER COEFFICIENT

Consider a unit cell of the ith type, in dimensionless form, Figure 1b. The limiting trajectory of a particle with radius a_p meets the entrance of a unit cell of the ith type at a distance $(r_l^*)_i$ from the axis (point K in Figure 1b). Let $(r_0^*)_i$ be the radius at the entrance of the unit cell beyond which the center of the particle cannot go due to the presence of the solid wall (Figure 1b). Of course, $(r_0^*)_i$ is a function of the particle radius and can be determined by a simple geometric calculation, (see Supplement).

All trajectories that result in capture in a unit cell of the ith type originate from a position at the entrance which has radial coordinate larger than or equal to $(r_l^*)_i$ and less than or equal to $(r_0^*)_i$. Based on this result, one can calculate the fraction impacted for a cell of the ith type, in other words, the number fraction of the particles which enter the unit cell per unit time, that come to collision with the wall, provided that one knows the particle concentration profile at the entrance. Let $C_i(1/2, r^*)$ be the volume fraction of the suspended particles at the entrance (such that $C_i[1/2, (r_0^*)_i] = 0$).

It will be assumed that the volume fraction at the entrance of a unit cell of the ith type is uniform for $0 \leq r^* \leq (r_0^*)_i$ and zero for $(r_0^*)_i < r^* \leq r_2^*$ (see Figure 1b). It will, further, be assumed that the non-zero value of $C_i(1/2, r^*)$ is the same for all unit cells belonging to the same unit bed element. This assumption of a flat concentration profile is made mainly for convenience in the absence of more suitable information. The assumption that $C_i(1/2, r^*)$ is the same for all unit cells of the same unit bed element implies perfect radial mixing in the packed bed, an assumption which was shown to be a valid one by Ison and Ives (1969) who demonstrated that during the initial filtration period the distribution of C, in the axial direction obeys the negative exponential law, in accordance with the filtration rate equation, Equation (4). Based on that assumption one obtains

$$\eta_{0i} = \frac{\int_{(r_i^*)_i}^{(r_0^*)_i} r^* v_z^* \left(\frac{1}{2}, r^*\right) C_i\left(\frac{1}{2}, r^*\right) dr^*}{\int_0^{(r_0^*)_i} r^* v_z^* \left(\frac{1}{2}, r^*\right) C_i\left(\frac{1}{2}, r^*\right) dr^*}$$

$$= \frac{\psi^*\left[\frac{1}{2}, (r_i^*)_i\right] - \psi^*\left[\frac{1}{2}, (r_0^*)_i\right]}{\psi^*\left(\frac{1}{2}, 0\right) - \psi^*\left[\frac{1}{2}, (r_0^*)_i\right]} \qquad (51)$$

The fraction impacted for the entire unit collector η_0 is given by

$$\eta_0 = \frac{\sum_{i=1}^{I_c} n_i(q_i - \tilde{q}_i)\eta_{0i}}{\sum_{i=1}^{I_c} n_i(q_i - \tilde{q}_i)} \qquad (52)$$

where q_i is the volumetric flow rate through a unit cell of the ith type and $\tilde{q}_i$ is the flow rate through the section of the entrance area of a unit cell of the ith type which corresponds to $(r_0^*)_i < r^* \leq r_2^*$. According to the P-T-T model, it is

$$q_i = \frac{v_s d_i^3}{N_c \langle d_c^3 \rangle} \quad i = 1, \ldots, I_c \qquad (53)$$

We also have

$$\tilde{q}_i = 2\pi\psi^*\left[\frac{1}{2}, (r_0^*)_i\right] h_i^2 (v_0)_i \qquad (54)$$

Substituting Equations (53) and (54) into Equation (52) one obtains

$$\eta_0 = \frac{\sum_{i=1}^{I_c} n_i \left\{\frac{v_s d_i^3}{N_c \langle d_c^3 \rangle} - 2\pi\psi^*\left[\frac{1}{2}, (r_0^*)_i\right] h_i^2 (v_0)_i\right\} \eta_{0i}}{\left(\frac{v_s}{N_c}\right) - 2\pi \sum_{i=1}^{I_c} n_i \psi^*\left[\frac{1}{2}, (r_0^*)_i\right] h_i^2 (v_0)_i} \qquad (55)$$

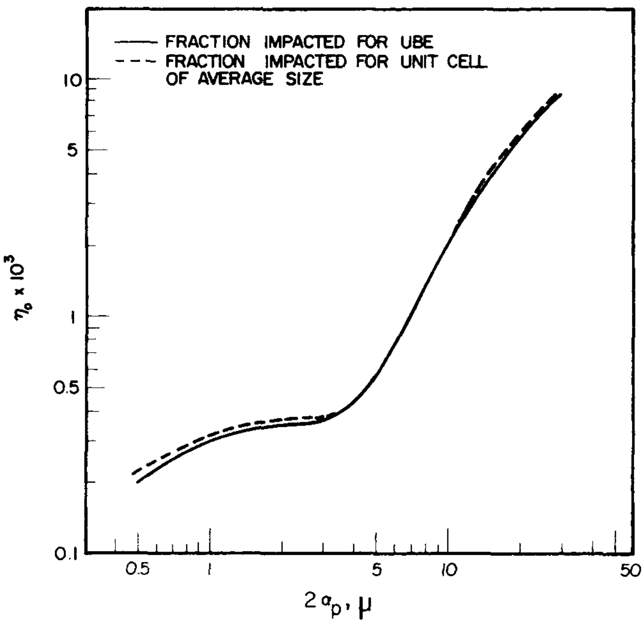

Fig. 5. Plot of fraction impacted for a unit bed element and for its unit cell of average size.

For small particles we have $\tilde{q}_i << q_i$, and then Equation (52) reduces to

$$\eta_0 = \frac{\sum_{i=1}^{I_c} n_i d_i^3 \eta_{0i}}{\langle d_c^3 \rangle} \qquad (56)$$

RATE OF DEPOSITION AS A FUNCTION OF POSITION ON THE UNIT CELL WALL

The filtration model developed herein can also be used to calculate the local rate of particle deposition on any point of the unit cell wall. Such information is valuable in estimating the geometrical changes of the void space of the packed bed caused by the deposition of particles, and thus it can be used to predict the effect of the specific deposit on the filtration efficiency and pressure gradient. An example of calculated rate of deposition as a function of position on the unit cell wall is given in the Supplement. It is shown that virtually all deposition occurs on the upper half of the unit cell surface.

DEPENDENCE OF η_{0i} ON THE SIZE OF THE UNIT CELL. SIMPLIFIED CALCULATION OF FILTER COEFFICIENT

It can be shown (see Supplement) that the calculated value of η_0 using Equation (55) and the corresponding value of η_{0i} for the average unit cell are considerably close. An example (corresponding to the conditions in Table 5 of Supplement) is given in Figure 5. A reasonable approximation would therefore be to calculate λ_0 using the value of η_{0i} corresponding to the unit cell of average size. This approximation results in a significant reduction in computation time, but it should not be expected to be valid for very large particles when a_p is comparable to the constriction radius of the smallest unit cell.

ACKNOWLEDGMENT

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NOTATION

A_c = cross section area of the bed
A_i, B_i, D_i = variables used for the description of the undisturbed flow in the immediate vicinity of the wall of a unit cell of the ith type, Equations (32)
A^*, B^*, D^* = dimensionless variables used for the description of the flow in the immediate vicinity of the wall of the dimensionless unit cell, defined by Equations (42) to (44)
A_i^+, B_i^+, D_i^+ = dimensionless variables defined by Equations (42) to (44)
a_i = maximum diameter of a unit cell of the ith type
a_p = suspended particle radius
$(a_p^*)_i = \frac{a_p}{h_i}$, dimensionless particle radius in a unit cell of the ith type
C = volume fraction of suspended particles
$C_i\left(\frac{1}{2}, r^*\right)$ = volume fraction profile at the entrance of a unit cell of the ith type
d_c = effective constriction diameter
d_g = grain diameter
d_i = constriction diameter of a unit cell of the ith type
$\{d_{i-1/2}; i = 1, \ldots, I_c + 1\}$ = constriction diameters cor-

responding to the set of suction values $\{p_{i-1/2}; i = 1, \ldots, I_c + 1\}$, see Payatakes et al. (1973a)

e = 4.8029×10^{-10} e.s.u. charge (1 e.s.u. charge = 1 cm dyne$^{1/2}$), elementary electric charge, that is, the charge of one electron

$\underline{F_D}$ = drag force exerted on the suspended particle

$\underline{F_E}$ = electrokinetic force exerted on the suspended particle

$\underline{F_G}$ = gravitational force exerted on the suspended particle

$\underline{F_I}$ = inertia force exerted on the suspended particle

$\underline{F_L}^{ret}$ = London force exerted on the suspended particle (corrected for the retardation effect)

$F_1(\delta^+)$, $F_2(\delta^+)$, $F_3(\delta^+)$, $F_4(\delta^+)$, $F_5(\delta^+)$, $F_6(\delta^+)$ = universal functions defined by Equations (35), (46) to (48), respectively (see also Supplement)

$f_P(\sigma; \underline{x}_3)$ = function expressing the effect of the specific deposit on the pressure gradient, Equation (5)

$f^m{}_{1x}(\delta^+)$, $f^m{}_{2x}(\delta^+)$, $f_x{}^r(\delta^+)$, $f_x{}^t(\delta^+)$ = universal functions, Equation (21) (and Supplement)

$f_y{}^m(\delta^+)$, $f_y{}^t(\delta^+)$ = universal functions, Equation (28) (and Supplement)

$f_\lambda(\sigma; \underline{x}_2)$ = function expressing the dependence of λ/λ_0 on σ, Equation (4)

$G_1(\delta^+)$, $G_2(\delta^+)$, $G_3(\delta^+)$ = universal functions defined by Equation (36) (see also Supplement)

g = gravitational acceleration

$g^m{}_{1w}(\delta^+)$, $g^m{}_{2w}(\delta^+)$, $g_w{}^r(\delta^+)$, $g_w{}^t(\delta^+)$ = universal functions, Equation (23) (and Supplement)

H = Hamaker constant for the system particle-medium-collector

h_i = height of a unit cell of the ith type

I_c = number of different types of unit cells

i = index

j = index

k = 1.38044×10^{-16} erg °K^{-1}, Boltzmann constant

L = total length of the filter bed

l = length of periodicity of the bed (Payatakes et al., 1973a)

m_j = concentration of the jth ionic species, number per cm^3

N_c = number of constrictions per unit area of a bed cross section; also, number of unit cells per unit area of a unit bed element (see Payatakes et al., 1973a)

N_{DL} = κa_p, double layer parameter (dimensionless)

N_{E1}, N_{E2} = first and second dimensionless electrokinetic groups defined as

$$\frac{\tilde{\epsilon}\kappa(\psi^2{}_{01} + \psi^2{}_{02})}{12\pi\mu v_s} \quad \text{and} \quad \frac{2\psi_{01}\psi_{02}}{(\psi^2{}_{01} + \psi^2{}_{02})},$$

respectively

$N_G = \dfrac{2(\rho_p - \rho)a_p{}^2 g}{9\mu v_s}$, dimensionless gravitational group

$N_{Lo} = \dfrac{H}{9\pi\mu a_p{}^2 v_s}$, London parameter (dimensionless)

$(N_{Re})_i = \dfrac{h_i(v_0)_i}{\nu}$, Reynolds number characterizing the flow through a unit cell of the ith type

$(N_{Re})_s = \dfrac{\langle d_g\rangle v_s}{\nu}$, superficial Reynolds number for a packed bed

$N_{Ret} = \dfrac{2\pi a_p}{\lambda_e}$, retardation parameter (dimensionless)

$N_{RS} = \dfrac{a_p}{\langle h\rangle} = \dfrac{a_p}{\langle d_g\rangle}$, relative size group (dimensionless)

n_i = number fraction of pores, the largest constrictions of which have values within the interval $[d_{i-1/2}, d_{i+1/2}]$; also, by assumption, the number fraction of unit cells of the ith type, (see Payatakes et al., 1973a)

P = pressure, including hydrostatic pressure

p = dimensionless length defined by Equation (30), used as argument of $\alpha_{s,p}$

$\{p_{i-1/2}; i = 1, \ldots, I_c + 1\}$ = set of arbitrarily chosen values of suction partitioning the entire region of interest of the initial drainage curve diagram (see Payatakes et al., 1973a)

Q = flow rate through the packed bed

q_i = flow rate through a unit cell of the ith type

$\tilde{q}_i$ = flow rate through the section of the entrance area of a unit cell of the ith type which corresponds to $(r_0{}^*)_i < r^* \leq r_2{}^*$; calculated from Equation (54)

r = radial cylindrical coordinate

$r^* = \dfrac{r}{h_i}$, dimensionless radial cylindrical coordinate

$(r_l{}^*)_i$ = dimensionless radial coordinate of the point of entrance of the limiting trajectory in a unit cell of the ith type, Figure 1b

$r_w(z)$ = distance of the unit cell wall from its axis at z

$r_w{}^*(z^*) = \dfrac{r_w(z)}{h_i}$, dimensionless distance of the unit cell wall from its axis at z^*

$(r_0{}^*)_i$ = dimensionless radial coordinate at the entrance of a unit cell of the ith type beyond which the center of a given particle cannot go due to the presence of the solid wall, Figure 1b

$r_1{}^*$ = dimensionless radius of the entrance constriction of the extended unit cell

$r_2{}^*$ = dimensionless maximum radius of the unit cell, Figure 1b

$r_3{}^*$ = dimensionless radius of the mid construction of the unit cell, Figure 1b

T = absolute temperature

$\underline{T_D}$, $\underline{T_E}$, $\underline{T_G}$, $\underline{T_I}$, $\underline{T_L}^{ret}$ = drag torque due to electrokinetics, gravitational, inertial and London forces, respectively

t = time measured from the startup of the filtration process

$\underline{u}$ = velocity of the center of the particle relative to a fixed point on the unit cell wall

u_x, u_w, u_y = x, w, y components of $\underline{u}$

$\underline{v}$ = fluid velocity in the absence of suspended particles (undisturbed fluid velocity)

v_s = superficial velocity Q/A_c

v_x, v_w, v_y = x, w, y components of $\underline{v}$

$(v_0)_i = \dfrac{q_i}{(r_1{}^* h_i)^2}$ = characteristic velocity for a unit cell of the ith type, (mean velocity at the entrance of the extended unit cell)

$\underline{v}^* = \dfrac{\underline{v}}{(v_0)_i}$, dimensionless fluid velocity vector (undisturbed)

w = coordinate tangent to the unit cell wall and normal to the plane $(z, r, 0)$, Figure 1a

x = coordinate tangent to the unit cell wall with origin at $(z^*, r^*, \theta) = (\frac{1}{2}, r_2{}^*, 0)$, Figure 1a

$\underline{x}_1, \underline{x}_2, \underline{x}_3$ = parameters vectors of $\lambda_0, f_\lambda, f_P$, respectively
y = coordinate normal to the unit cell wall, with origin on the wall, Figure 1a
z = axial cylindrical coordinate
z_j = valency of the jth ionic species
z_P = value of z at a certain point P on the unit cell wall (Figure 1a)
z^* = $\frac{z}{h_i}$, dimensionless axial cylindrical coordinate

Greek Letters

α = angle formed by the z-axis and the tangent on the wall, defined by Equation (11)
$\alpha_{s,p}$ = retardation factor for London, the force between a sphere and a plate with semi-infinite thickness
ΔP = pressure drop across the filter bed
δ = separation between spherical particle and unit cell wall, defined by Equation (29)
δ^+ = $\frac{\delta}{a_p}$, dimensionless separation
δ^+_{cr} = minimum dimensionless separation at which $F_{Ey} + F_{Ly}^{\text{ret}} = 0$
$\tilde{\delta}^+_{cr}(\zeta_1^+)$ = minimum dimensionless separation at which $F_{Ey} + F_{Ly}^{\text{ret}} + F_{Gy} + F_{Dy} = 0$, for a given ζ_1^+
ϵ = macroscopic porosity of the bed
ϵ_d = porosity of deposited matter
ϵ_0 = macroscopic porosity of the clean bed
$\tilde{\epsilon}$ = dielectric constant of liquid medium
ζ_1, ζ_2 = position coordinates used in trajectory calculation, see Figure 1a
ζ_1^*, ζ_2^* = defined as $\frac{\zeta_1}{h_i}$ and $\frac{\zeta_2}{h_i}$, respectively
ζ_1^+, ζ_2^+ = defined as $\frac{\zeta_1}{a_p}$ and $\frac{\zeta_2}{a_p}$, respectively
η_{BM0} = fraction impacted due to Brownian motion alone
η_{T0} = fraction impacted due to gravitational, hydrodynamic, electrokinetic, London, as well as Brownian forces
η_0 = particle number fraction impacted on a clean unit bed element
η_{0i} = fraction impacted in a unit cell of the ith type
$\hat{\eta}_0$ = particle number fraction collected by a clean unit bed element
θ = angular cylindrical coordinate
κ = double layer reciprocal thickness, Equation (31)
λ = filter coefficient
λ_e = wave length of electron oscillation
λ_{T0} = $\frac{1}{l}\ln\left(\frac{1}{1-\eta_{T0}}\right)$ filter coefficient for a clean bed taking into account gravitational, hydrodynamic, electrokinetic, London, as well as Brownian forces
λ_0 = filter coefficient for a clean bed
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density of liquid
ρ_p = suspended particle density
σ = specific deposit (volume of deposited matter per unit bed volume)
τ = time measured from the moment at which the suspension first reached the bed position under consideration
$\Phi(z^*)$ = function defined by Equation (13)
Φ_P = quantity defined by Equation (15)
Φ_0 = quantity defined by Equation (16)
ϕ_s = shape factor of the grains composing the packed bed
ψ = stream function
ψ_{01}, ψ_{02} = surface potential of suspended particle and grain, respectively; (the surface potential is approximately equal to the zeta potential, and for this reason the value of the latter, which can be determined experimentally, is usually used)
ψ^* = $\frac{\psi}{h_i^2(v_0)_i}$ = dimensionless stream function
ω_w = angular velocity of suspended (spherical) particle

Subscripts

w = component in the w direction
x = component in the x direction
y = component in the y direction

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Part II. Case Study of the Effect of the Dimensionless Groups and Comparison with Experimental Data

A case study is made to determine the dependence of the deep bed filtration rate (expressed in terms of the fraction of suspended particles impacted) on the eight dimensionless parameters, which are found to be relevant in the filtration process, based on the trajectory calculation method developed in Part I of this series. In addition, comparisons between results based on the theoretical model of this work and available experimental data are made. Comparisons are also made with some of FitzPatrick's theoretical results. The results of this study demonstrate clearly the complex and interactive nature of the relation between the various parameters and the efficiency of filtration. Accordingly, the conventional format of correlating experimental data, which equates the filter coefficient with a product of the pertinent dimensionless groups, each raised to an empirical exponent, will not be adequate in providing a generalized correlation of experimental filtration data.

SCOPE

The use of dimensional analysis for correlating deep bed filtration data was made rather recently by Ison and Ives (1969) in an attempt to establish the dependence of the initial filter coefficient on a few of the dimensionless parameters pertinent to deep bed filtration. More recently FitzPatrick (1972) proposed a theoretical model of deep bed filtration, based on which he calculated the dependence of the initial filter coefficient on some of the dimensionless groups appearing in his mathematical formulation.

In this part of this series, the theoretical model of deep bed filtration and the method for its solution developed in Part I are used to make a more complete study of the dependence of the fraction impacted on the dimensionless parameters appearing in the mathematical formulation. Apart from the fact that the porous media model used in the present study (P-T-T model) is more realistic than models used by previous investigators, the present study is also more complete since it includes the retardation effect on the molecular dispersion force.

This study concludes with comparisons between theoretical results and experimental data, as well as comparisons with some of FitzPatrick's theoretical results.

CONCLUSIONS AND SIGNIFICANCE

The dependence of the fraction (of suspended particles) impacted, η_0, on each of the dimensionless groups appearing in the trajectory equation is determined in a case study. It is found that when the molecular dispersion force dominates the double layer interaction force at all separations the effect of the latter is negligible, but a sharp drop of η_0 (or of the initial filter coefficient, λ_0) in excess of three orders of magnitude is observed for any change of the parameters that leads to a situation in which a repulsive double layer interaction force becomes sufficiently larger in magnitude than the attractive molecular dispersion force at separations larger than a critical one. Such situations may take place through a combination of several factors. The dependence of η_0, (or λ_0) on the various dimensionless groups is such that the quantity $\lambda_0 \langle d_g \rangle$ cannot be expressed as a product of the powers of the various dimensionless groups over extended ranges of these groups, and the simple product expression suggested by Ison and Ives (1969) has only limited validity over small ranges of the dimensionless groups. For studies over wide ranges of the group values, the designer has to resort to additional experimentation or to